An Intransitivity Model for Matchup and Pairwise Comparison

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Objectives

- Predict pairwise matchup.
- Unify several existing models on representation learning.

Intransitivity

An intuitive example a rock-paper-scissors game, the pairwise matchup result is judged by three rules: $\{paper \succ rock$, $rock \succ scissors$ and $scissors \succ paper$ $\}$. A transitive model results in a transitive dominance $paper \succ scissors$, that violates the third rule $scissors \succ paper$.

Examples sports tournaments; online games; election process.

Related works

Problem setting

- Candidate players i and $i \in \mathbf{P}$ with $|\mathbf{P}| = N$.
- 4-tuples (i, j, n_i, n_j), and $i \succ j := (i, j, 1, 0)$.
- Matchup matrix $\mathbf{M} \in \mathcal{R}^{N \times N}$.
- $M_{ij} > 0$ means item i has a comparative advantage over item j.

1. Bradley-Terry Model (Bradley et al., 1952)

$$\Pr(i \succ j) = \frac{\exp(\gamma_i)}{\exp(\gamma_i) + \exp(\gamma_j)} = \frac{1}{1 + \exp(-M_{ij}))}, \tag{1}$$

where γ_i is the ability of winning for player i.

$$M_{ij} = \gamma_i - \gamma_j.$$

2. Blade-Chest-Inner Model (Chen et al., 2016a)

$$M_{ij} = \mathbf{x}_i^{\text{blade}\top}\mathbf{x}_j^{\text{chest}} - \mathbf{x}_j^{\text{blade}\top}\mathbf{x}_i^{\text{chest}} + \gamma_i - \gamma_j, \tag{2}$$

3. Neural network framework of Blade-Chest (Chen et al., 2016b)

Top layer is the blade-chest-inner model (2):

$$\Pr(i \succ j|g) = \sigma(M(i, j|g)).$$

4. Blade-Chest-Sigma Model (Duan et al., 2017)

$$M_{ij} = \mathbf{x}_i^\top \Sigma \mathbf{x}_j + \mathbf{x}_i^\top \Gamma \mathbf{x}_i - \mathbf{x}_j^\top \Gamma \mathbf{x}_j, \tag{3}$$

where $\Sigma, \Gamma \in \mathcal{R}^{d \times d}$ are the transitive matrices.

Proposed method

- A generalized model with more expressiveness by a low-rank matrix.
- Neural Network framework of the generalized model.
- A quantitative evaluation of the existence of intransitivity in datasets.

Matchup function

Define the representation matrix as

$$\mathbf{X}^{\text{blade}} = (\mathbf{x}_1^{\text{blade}}, \dots, \mathbf{x}_N^{\text{blade}}), \quad \mathbf{X}^{\text{chest}} = (\mathbf{x}_1^{\text{chest}}, \dots, \mathbf{x}_N^{\text{chest}}), \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_N).$$

$$(2) \Rightarrow \mathbf{M} = \mathbf{X}^{\text{blade}\top}\mathbf{X}^{\text{chest}} - \mathbf{X}^{\text{chest}\top}\mathbf{X}^{\text{blade}} + \boldsymbol{\gamma}^\top \mathbf{1} - \mathbf{1}^\top \boldsymbol{\gamma}.$$

Replace the matrix product $\mathbf{X}^{\text{blade}\top}\mathbf{X}^{\text{chest}}$ by a new matrix $\mathbf{Y}$ as

$$\mathbf{Y} = \mathbf{X}^{\text{blade}\top}\mathbf{X}^{\text{chest}},$$

which results in a general representation of the matchup matrix as

$$\mathbf{M} = \left(\boldsymbol{\gamma}^\top \mathbf{1} - \mathbf{1}^\top \boldsymbol{\gamma}\right) + \left(\mathbf{Y} - \mathbf{Y}^\top\right) \ \text{ s.t. } \operatorname{rank}(\mathbf{Y}) \le D. \tag{4}$$

Objection (assuming i is the winner; same with MSE loss)

$$_{\Theta} \Sigma_{(i,j)} \log \Pr(i \succ j|\Theta). \tag{5}$$

Top layer the generalized intransitivity model (4):

$$\Pr(i \succ j) = \sigma(M_{ij}) = \sigma(Y_{ij} - Y_{ji}).$$

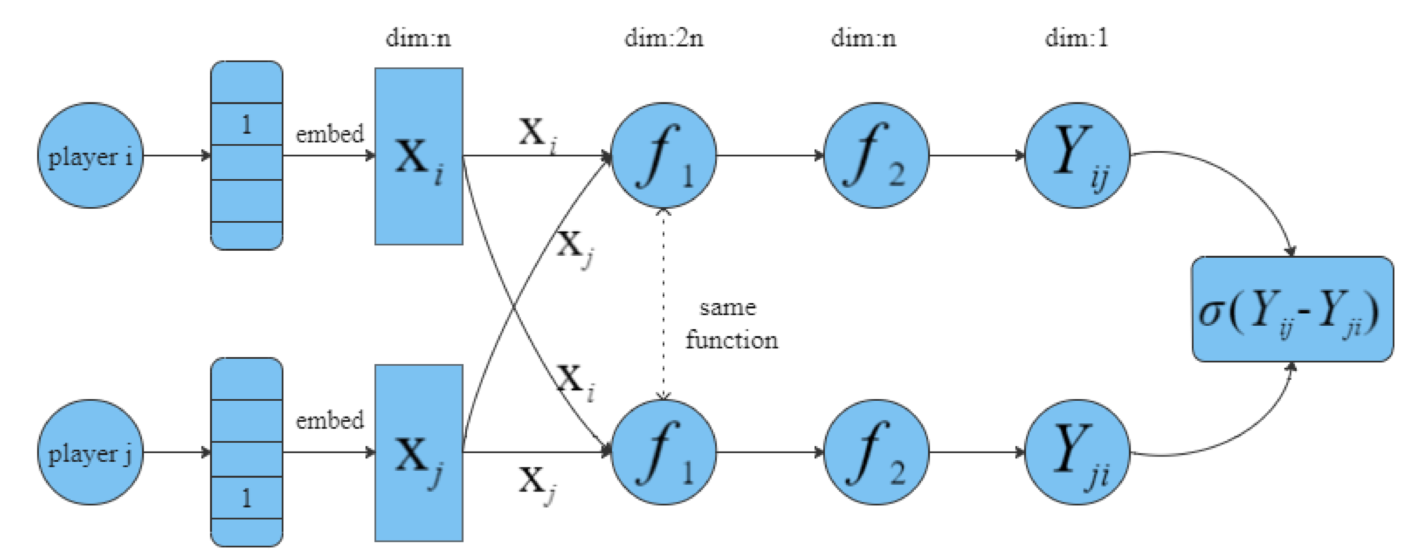

Figure: An illustration of the proposed generalized intransitivity framework

Properties

1. $\mathbf{Y} - \mathbf{Y}^\top$ can represent an arbitrarily complex matchup matrix by removing the rank constraint.
2. It is equivalent to the BT model when $\operatorname{rank}(\mathbf{Y}) = 0$.
3. It can represent the intransitivity when $\operatorname{rank}(\mathbf{Y}) = 1$:

$$\mathbf{Y} = (x_1^{\text{blade}}, x_2^{\text{blade}}, \dots, x_N^{\text{blade}})^\top (x_1^{\text{chest}}, x_2^{\text{chest}}, \dots x_N^{\text{chest}}),$$

the matchup matrix without the strength terms becomes

$$M_{ij} = x_i^{\text{blade}} x_j^{\text{chest}} - x_i^{\text{chest}} x_j^{\text{blade}}.$$

Assume that $i \succ j$ and $j \succ k$ (i.e., $M_{ij} > 0$ and $M_{jk} > 0$), then taking $x_i^{\text{chest}} > 0$, $x_j^{\text{chest}} < 0$, and $x_k^{\text{chest}} > 0$ shows $k \succ i$ (i.e., $M_{ik} < 0$).

Datasets and Experiments

- **Intrans.**: whether the dataset contains intransitivity
- **No.IntPlayer**: the fraction of players invovled in rock-paper-scissors
- **Int.Ratio**: the amount of the rock-paper-scissors-like relationship (%)

Table: Summary of the real-world datasets

Dataset	Players	Records	Intrans.	No.IntPlayer	Int.Ratio
SushiA	10	100000	no	0	0
SushiB	100	25000	yes	92	26.87%
MovieLens100K	1682	139982	yes	1130	0.19%
Election A5	16	44298	yes	6	0.44%
Election A9	12	95888	yes	5	1.82%
Election A17	13	21037	yes	8	8.18%
Election A48	10	25848	no	0	0
Election A81	11	44298	yes	5	2.50%
SF4-5000	35	5000	yes	34	23.86%
Dota	757	10442	yes	550	97.58%
Pokemon	800	50000	yes	784	78.58%

How does the proposed method perform?

Table: Test accuracy on the real-world datasets

Dataset	Bradley-Terry	Blade-Chest-Inner	Blade-Chest-Sigma	Neural BC	Proposed model
SushiA	0.6525 ± 0.0011	0.6546 ± 0.0006	0.6560 ± 0.0004	0.6630 ± 0.0004	$\mathbf{0.6632 \pm 0.0003}$
SushiB	0.6257 ± 0.0025	0.6235 ± 0.0150	0.6414 ± 0.0019	0.6561 ± 0.0017	$\mathbf{0.6563 \pm 0.0011}$
MovieLens100K	0.6785 ± 0.0005	0.6792 ± 0.0004	0.6789 ± 0.0003	0.6950 ± 0.0019	$\mathbf{0.6973 \pm 0.0002}$
Election A5	0.6478 ± 0.0017	0.6489 ± 0.0011	0.6494 ± 0.0018	0.6550 ± 0.0030	$\mathbf{0.6560 \pm 0.0018}$
Election A9	0.6028 ± 0.0003	0.6096 ± 0.0007	0.6047 ± 0.0008	0.6174 ± 0.0003	$\mathbf{0.6175 \pm 0.0003}$
Election A17	0.5189 ± 0.0001	0.5305 ± 0.0010	0.5296 ± 0.0013	0.5582 ± 0.0003	$\mathbf{0.5598 \pm 0.0002}$
Election A48	0.5993 ± 0.0001	0.6001 ± 0.0001	0.5996 ± 0.0001	0.6060 ± 0.0001	$\mathbf{0.6056 \pm 0.0001}$
Election A81	0.6013 ± 0.0001	0.6018 ± 0.0001	0.6011 ± 0.0002	0.6194 ± 0.0001	$\mathbf{0.6194 \pm 0.0001}$
SF4-5000	0.5079 ± 0.0078	0.5181 ± 0.0171	0.5358 ± 0.0049	0.5514 ± 0.0008	$\mathbf{0.5496 \pm 0.0021}$
DotA	0.6334 ± 0.0077	0.6432 ± 0.0034	0.6420 ± 0.0051	0.6468 ± 0.0031	$\mathbf{0.6485 \pm 0.0025}$
Pokemon	0.8157 ± 0.0094	0.8495 ± 0.0016	0.8187 ± 0.0168	0.8943 ± 0.0040	$\mathbf{0.8949 \pm 0.0021}$

References

Bradley et al., 1952, Ran analysis of incomplete block designs: I. the method of paired comparisons.
Chen et al., 2016a, Modeling intransitivity in matchup and comparison data.
Chen et al., 2016b, Predicting matchups and preferences in context.
Duan et al., 2017, A Generalized Model for Multidimensional Intransitivity.