ICPR 2020
25th INTERNATIONAL CONFERENCE ON PATTERN RECOGNITION
ONLINE 10 | 15 January 2021

Exact and Convergent Iterative Methods to Compute the Orthogonal Point-to-Ellipse Distance

Siyu Guo*, Pingping Hu, Zhigang Ling, He Wen, Min Liu, Lu Tang

College of Electrical and Information Engineering, Hunan University, Changsha, China

*Email: syguo75@163.com

The orthogonal distance provides a natural and undistorted measure to evaluate how close a point is to a given ellipse, and is thus useful for accurate ellipse fitting. This paper presents an exact algorithm and a convergent iterative algorithm to compute the orthogonal distance between a point and an ellipse.

1. Problem formulation

Since the orthogonal distance from a point to an ellipse is translation- and rotation-invariant, without loss of generality, the ellipse can be relocated, through translation and rotation and along with the point, as to be centered at the origin with the majoraxis lying along the x-axis. The transformed ellipse is then of the following simple equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

It can be easily shown that the point under concern, $(x'_{\mathrm{O}}, y'_{\mathrm{O}})$, and its orthogonal contacting point on the ellipse, $(x'_{\mathrm{E}}, y'_{\mathrm{E}})$, are in a same quadrant. Suppose $(x'_{\mathrm{O}}, y'_{\mathrm{O}})$ is in the first quadrant without loss of generality, let the orthogonal contacting point be expressed as

$$x'_{\mathrm{E}}=a\cos\varphi,\ y'_{\mathrm{E}}=b\sin\varphi,\ \varphi\in[0,\pi/2] \tag{2}$$

The orthogonal contacting point can be determined by minimizing the distance function

$$D(\varphi)=\frac{1}{2}\left[(x'_{\mathrm{O}}-a\cos\varphi)^2+(y'_{\mathrm{O}}-b\sin\varphi)^2\right] \tag{3}$$

which can in turn be solved by finding the nonnegative root of the following quartic equation

$$\alpha_4 s^4+\alpha_3 s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0=0 \tag{4}$$

where $s = \sin\varphi$,

$$\begin{cases}\alpha_4=-d^2\\ \alpha_3=-\alpha_1=-2y'_{\mathrm{O}}bd\\ \alpha_2=d^2-\left(a^2x'^2_{\mathrm{O}}+b^2y'^2_{\mathrm{O}}\right)\\ \alpha_0=b^2y'^2_{\mathrm{O}}\end{cases}$$

and $d = a^2 - b^2$.

2. The exact algorithm

The closed-form solution to Eq. (4) is readily available, which is provided in the paper and omitted in the poster for clarity.

However, the closed-form solution has two major drawbacks. It is computationally expensive, and due to the numeric errors during the computation, the algorithm may fail to give the desired nonnegative real root, although the latter problem can be bypassed by extra examinations and comparisons.

3. The convergent iterative algorithm

Eq. (4) can also be solved numerically through Newton's method

$$s_{k+1}=\frac{\left((3\alpha_1 s_k+2\alpha_2)s_k+\alpha_3\right)s_k^2-\alpha_5}{\left((4\alpha_1 s_k+3\alpha_2)s_k+2\alpha_3\right)s_k+\alpha_4} \tag{5}$$

It is proven in the paper that from the initial solution $s_0 = 1$, the iteration (5) converges to the desired solution of Eq. (4). This can also be intuitively seen in Fig. 1, where different curves correspond to different conditions exhaustively enumerated in the paper.

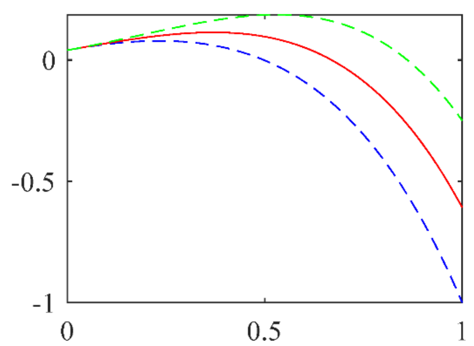

Fig. 1 Curves of the function in (4) under different conditions

4. Experimental results

The exact algorithm (EA) and convergent iterative algorithm (CA) are compared with the one proposed by Ahn et al. (ARW01) in experiments.

In one experiment, distances from 2001×2001 grid points in a square region to an ellipse in the region are computed, and the results are listed in Table I. In another experiment, 100 ellipses are randomly generated, and for each ellipse, 100 to 1000 random points are subsequently generated, and the orthogonal distances are computed. The results are listed in Table II.

In both tables, r_{SD} is the proportion that the algorithm yields the shortest distance, r_{AF} is the proportion of failure to give a solution, and d_{MO} is the maximal difference between the algorithm outcomes and the orthogonal distances.

TABLE I. COMPARISON OF ALGORITHM PERFORMANCES IN THE GRID POINTS EXPERIMENT

Algorithm	**ARW01**	**EA**	**CA**
Exe. Time (s)	2.463	12.054	0.967
r_{SD}	0.55	0.41	0.41
r_{AF}	0.45	0	0
d_{MO}	2.2×10^{-6}	0.49	2.2×10^{-6}

TABLE II. COMPARISON OF ALGORITHM PERFORMANCES IN THE RANDOM ELLIPSES EXPERIMENT

Algorithm	**ARW01**	**EA**	**CA**
Exe. Time (s)	0.008	0.258	0.017
r_{SD}	0.47	0.32	0.27
r_{AF}	0.01	0	0
d_{MO}	9.5×10^{-6}	6.3×10^{-6}	6.3×10^{-6}

It can be seen from the tables that although ARW01 achieves the shortest distances in more cases, it encounters divergent iterations in both experiments. Since the random ellipses experiment can be soundly seen as similar to real scenarios, this implies that the use of ARW01 is likely to lead to failures caused by such divergence.

On the other hand, the distances given by EA and CA are often longer than the orthogonal ones, but for CA the differences are very minute. No algorithm failure is reported, verifying the theoretic analysis results.

As for the execution speed, CA and ARW01 are much faster than EA.

In a nutshell, the convergent iterative algorithm is a fast, accurate and reliable method for the computation of orthogonal distances between points and ellipses.