An Intransitivity Model for Matchup and Pairwise Comparison

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Outline

Intransitivity

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- $\Pr(i \succ j) > 0.5$
- $\Pr(j \succ k) > 0.5$
- $\Pr(k \succ i) > 0.5$.

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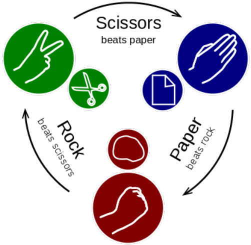
Examples:

- rock-paper-scissor game;
- sports tournaments;
- online games;
- election process;
- etc..



Picture from Wikipedia By Enzoklop

Existing Methods: Bradley-Terry Model [1, 2]

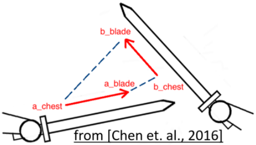
- transitive;
- the strength of each player is represented by a single real number γ;

$$\begin{aligned} \Pr(i \succ j) &= \frac{\exp(\gamma_i)}{\exp(\gamma_i) + \exp(\gamma_j)} \\ &= \frac{1}{1 + \exp\left(-(\gamma_i - \gamma_j)\right)} \\ &= \sigma(M_{ij}). \end{aligned} \tag{1}$$

- $M_{ij} = \gamma_i - \gamma_j$ is the matchup function of player i and j;
- $M_{ij} = -M_{ji} \quad \Rightarrow \quad \Pr(i \succ j) = 1 - \Pr(j \succ i)$ (negative symmetry).

Existing Methods: Blade-Chest Model [3]

- The Blade-Chest model allows intransitive relations by introducing two extra D-dimensional vectors for each player i: strength: 'Blade'; weakness: 'Chest';



- matchup function: the Blade-Chest-Inner model

$$M_{ij} = \mathbf{x}_i^{\text{blade}\top} \mathbf{x}_j^{\text{chest}} - \mathbf{x}_j^{\text{blade}\top} \mathbf{x}_i^{\text{chest}} + \gamma_i - \gamma_j, \tag{2}$$

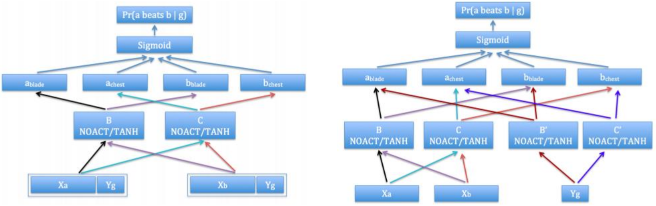
Neural Network Blade-Chest Framework [4]

Maximize the log-likelihood on the training dataset: (assuming i is the winner)

$$\arg\max_{\Theta} \ \Sigma_{(i,j,g)} \log \Pr(i \succ j|\Theta, g),$$

where Θ denotes all the parameters. Top layer is the blade-chest-inner model (2):

$$\Pr(i \succ j|g) = \sigma(M(i,j|g)).$$



(a) CONCAT model

(b) SPLIT model

Figure 1: Neural network framework of Blade-Chest model (game feature vectors contained)

Proposed Method: Low-rank Matrix Decomposition

Define the representation matrix as

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}^{\text{blade}} \\ \mathbf{X}^{\text{chest}} \end{pmatrix}, \quad \mathbf{X}^{\text{blade}} = (\mathbf{x}_1^{\text{blade}}, \ldots, \mathbf{x}_N^{\text{blade}}), \quad \mathbf{X}^{\text{chest}} = (\mathbf{x}_1^{\text{chest}}, \ldots, \mathbf{x}_N^{\text{chest}}).$$

We also denote the strength parameters in the original Bradley-Terry model by

$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N).$$

The Blade-Chest-Inner model (2) can be represented as

$$\mathbf{M} = {\mathbf{X}^{\text{blade}}}^\top \mathbf{X}^{\text{chest}} - {\mathbf{X}^{\text{chest}}}^\top \mathbf{X}^{\text{blade}} + \boldsymbol{\gamma}^\top \mathbf{1} - \mathbf{1}^\top \gamma. \tag{3}$$

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The Blade-Chest-Inner model (2) can be represented as

$$\mathbf{M} = {\mathbf{X}^{\text{blade}}}^\top \mathbf{X}^{\text{chest}} - {\mathbf{X}^{\text{chest}}}^\top \mathbf{X}^{\text{blade}} + \boldsymbol{\gamma}^\top \mathbf{1} - \mathbf{1}^\top \boldsymbol{\gamma}. \tag{3}$$

Replace the matrix product ${\mathbf{X}^{\text{blade}}}^\top \mathbf{X}^{\text{chest}}$ by a new matrix $\mathbf{Y}$ as

$$\mathbf{Y} = {\mathbf{X}^{\text{blade}}}^\top \mathbf{X}^{\text{chest}},$$

which results in a general representation in terms of low-rank matrix decomposition as

$$\mathbf{M} = \left(\boldsymbol{\gamma}^\top \mathbf{1} - \mathbf{1}^\top \boldsymbol{\gamma}\right) + \left(\mathbf{Y} - \mathbf{Y}^\top\right) \quad \text{s.t.} \quad \text{rank}(\mathbf{Y}) \le D. \tag{4}$$

Represent the Existing Models as Special Cases

1. $\mathbf{Y} - \mathbf{Y}^\top$ can represent an arbitrarily complex matchup matrix by removing the rank constraint;
2. It can represent the intransitivity when $\mathrm{rank}(\mathbf{Y}) = 1$:

$$\mathbf{Y} = (x_1^{\text{blade}}, x_2^{\text{blade}}, \ldots, x_N^{\text{blade}})^\top (x_1^{\text{chest}}, x_2^{\text{chest}}, \ldots x_N^{\text{chest}}),$$

   the matchup matrix without the strength terms becomes

$$M_{ij} = x_i^{\text{blade}} x_j^{\text{chest}} - x_i^{\text{chest}} x_j^{\text{blade}}.$$

   Assume that $i \succ j$ and $j \succ k$ (i.e., $M_{ij} > 0$ and $M_{jk} > 0$), then taking $x_i^{\text{chest}} > 0$, $x_j^{\text{chest}} < 0$, and $x_k^{\text{chest}} > 0$ shows $k \succ i$ (i.e., $M_{ik} < 0$).
3. It is equivalent to the BT model when $\mathrm{rank}(\mathbf{Y}) = 0$.

Framework of the Generalized Intransitivity Model

Objective function: (assuming i is the winner; same with MSE loss)

$$\arg\max_{\Theta} \ \Sigma_{(i,j)} \log \Pr(i \succ j | \Theta). \tag{5}$$

Top layer is the generalized intransitivity model (4):

$$\Pr(i \succ j) = \sigma(M_{ij}) = \sigma(Y_{ij} - Y_{ji}).$$

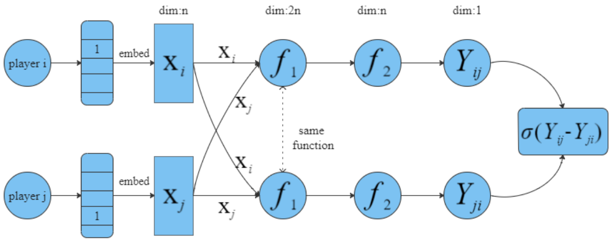

Figure 2: An illustration of the proposed generalized intransitivity framework (simpler)

Experiment Setup

- Competitive methods:
  1. Bradley-Terry model (BT model): transitive;
  2. Blade-Chest-Inner model: intransitive;
  3. Blade-Chest-Sigma model [5]: intransitive;

$$M_{ij} = \mathbf{x}_i^\top \Sigma \mathbf{x}_j + \mathbf{x}_i^\top \Gamma \mathbf{x}_i - \mathbf{x}_j^\top \Gamma \mathbf{x}_j, \tag{6}$$

     where $\Sigma, \Gamma \in \mathcal{R}^{d \times d}$ are the transitive matrices.
  4. Blade-Chest-Inner with neural network framework (Neural BC).
- The test accuracy is defined as

$$A(\mathcal{D}'|\mathbf{Y}) = \frac{1}{N'} \sum_{(i,j) \in \mathcal{D}} \mathbf{1}(i \succ j),$$

where N' is the total number of games in the testing set.

Experiments with Real-world Datasets

Table 1: Summary of the real-world datasets

Dataset	Players	Records	Intrans.	No.IntPlayer	Int.Ratio
SushiA	10	100000	no	0	0
SushiB	100	25000	yes	92	26.87%
MovieLens100K	1682	139982	yes	1130	0.19%
Election A5	16	44298	yes	6	0.44%
Election A9	12	95888	yes	5	1.82%
Election A17	13	21037	yes	8	8.18%
Election A48	10	25848	no	0	0
Election A81	11	44298	yes	5	2.50%
SF4-5000	35	5000	yes	34	23.86%
Dota	757	10442	yes	550	97.58%
Pokemon	800	50000	yes	784	78.58%

Results of Real-world Datasets

Table 2: Test accuracy on the real-world datasets

Dataset	Bradley-Terry	Blade-Chest-Inner	Blade-Chest-Sigma	Neural BC	Proposed model
SushiA	0.6525 ± 0.0011	0.6546 ± 0.0006	0.6560 ± 0.0004	0.6630 ± 0.0004	$\mathbf{0.6632 \pm 0.0003}$
SushiB	0.6257 ± 0.0025	0.6235 ± 0.0150	0.6414 ± 0.0019	0.6561 ± 0.0017	$\mathbf{0.6563 \pm 0.0011}$
MovieLens100K	0.6785 ± 0.0005	0.6792 ± 0.0004	0.6789 ± 0.0003	0.6950 ± 0.0019	$\mathbf{0.6973 \pm 0.0002}$
Election A5	0.6478 ± 0.0017	0.6489 ± 0.0011	0.6494 ± 0.0018	0.6550 ± 0.0030	$\mathbf{0.6560 \pm 0.0018}$
Election A9	0.6028 ± 0.0003	0.6096 ± 0.0007	0.6047 ± 0.0008	0.6174 ± 0.0003	$\mathbf{0.6175 \pm 0.0003}$
Election A17	0.5189 ± 0.0001	0.5305 ± 0.0010	0.5296 ± 0.0013	0.5582 ± 0.0003	$\mathbf{0.5598 \pm 0.0002}$
Election A48	0.5993 ± 0.0001	0.6001 ± 0.0001	0.5996 ± 0.0001	0.6060 ± 0.0001	$\mathbf{0.6056 \pm 0.0001}$
Election A81	0.6013 ± 0.0001	0.6018 ± 0.0001	0.6011 ± 0.0002	0.6194 ± 0.0001	$\mathbf{0.6194 \pm 0.0001}$
SF4-5000	0.5079 ± 0.0078	0.5181 ± 0.0171	0.5358 ± 0.0049	0.5514 ± 0.0008	$\mathbf{0.5496 \pm 0.0021}$
DotA	0.6334 ± 0.0077	0.6432 ± 0.0034	0.6420 ± 0.0051	0.6468 ± 0.0031	$\mathbf{0.6485 \pm 0.0025}$
Pokemon	0.8157 ± 0.0094	0.8495 ± 0.0016	0.8187 ± 0.0168	0.8943 ± 0.0040	$\mathbf{0.8949 \pm 0.0021}$

Conclusion

1. A generalized intransitivity model by a low-rank matrix approach.
2. A simple neural network framework for modeling the matchup and pairwise comparison;
3. Unify the existing methods;
4. A quantitative investigation of intransitivity in real-world datasets;
5. Effective for modeling the intransitive relationships;
6. Promising predictive performance.

References

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Thanks for your attention!