Stabilized Calculation of Gaussian Smoothing and Its Differentials Using Attenuated Sliding Fourier Transform

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1 Introduction

1.1 Gaussian smoothing and its first and second differentials

1.1.1 Applications

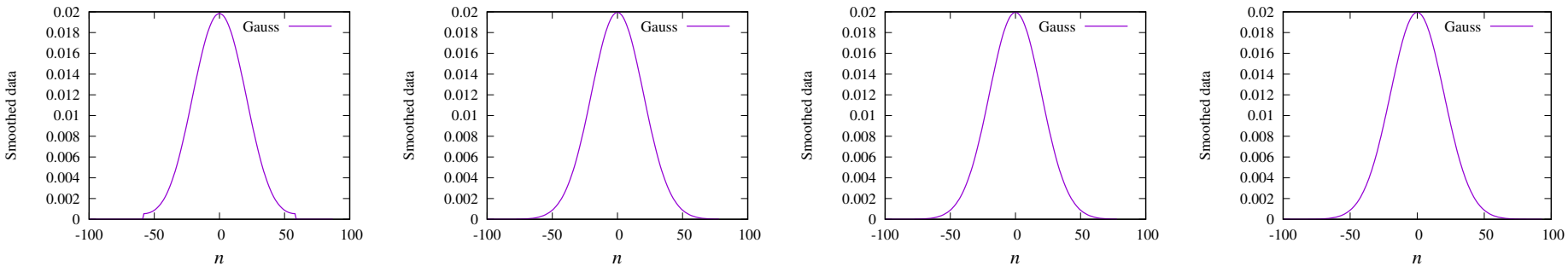
- Feature extraction
- Region based image matching
- Pattern recognition
- Scale space method

1.1.2 Computational complexity

- $M \times N$: Image size
- σ : scale of smoothing
- Direct calculation: $O(\sigma MN)$
  - Computational complexity is **proportional to σ**.
- FFT: (# of pixels) $O(MN(\log_2 M + \log_2 N))$
  - Calculation of FFT is **complex and slow**.
- To reduce it, the sub-sampling technique is used in SIFT.

1.2 Fast calculation of Gaussian smoothing

- By selecting the length of transformation properly, Gaussian function can be approximated by **a small number of sinusoidal functions**.
- Elboher et al. proposed to use **kernel integral**.
- Sugimoto et al. examined various types of **sliding DCTs** and recommended using the sliding DCT-1 for Gauss smoothing.
- Yamashita and Wakahara proposed an object detection method using Gaussian smoothing by **sliding Fourier transform** (SFT).

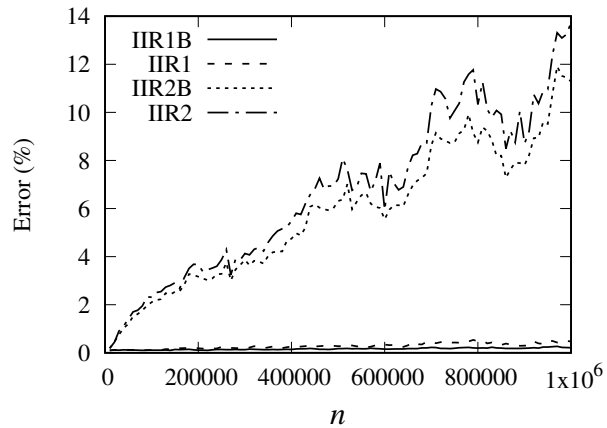


(a) $P = 2$ (b) $P = 3$ (c) $P = 4$ (d) $P = 6$

Approximated Gaussian function using SFT ($\sigma = 20.0$)

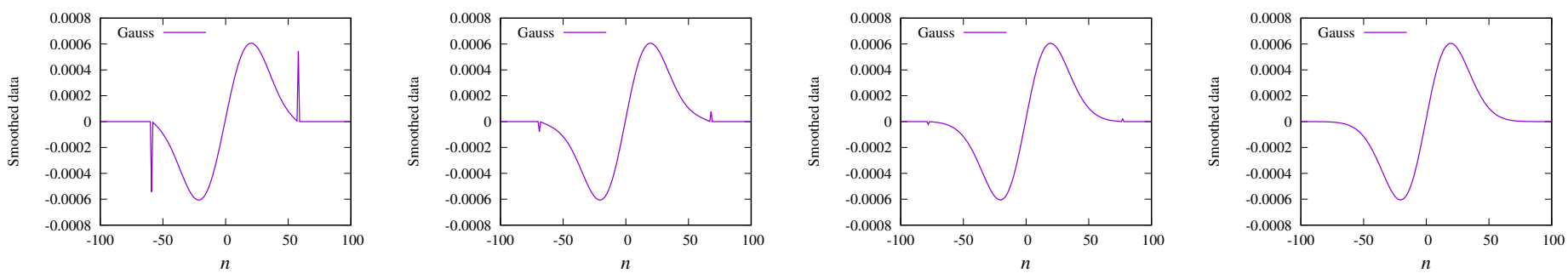
1.3 Problem

- **Error accumulation** by limited precision of floating-point calculation.



RMSE by single precision floating-point calculation (P=3)

- **Discontinuities at edges** of an interval for calculation of SFT.



(a) $P = 2$ (b) $P = 3$ (c) $P = 4$ (d) $P = 6$

False extreme points of Gaussian smoothed image ($\sigma = 20.0$) of input signal

$$x[n] = \begin{cases} -1 & (n = -1) \\ 1 & (n = 0) \\ 0 & (\text{else}) \end{cases}$$

1.4 Purposes of this research

- To propose the **attenuated SFT (ASFT)** by introducing a decay factor to SFT and to apply to Gaussian smoothing.
- To propose a criterion of **minimum mean square error with the aim of suppressing discontinuities at edges**.
- Show their advantages by experiments.

2 Gaussian smoothing and its differentials using SFT

2.1 Definitions

- Gaussian function and its differentials ($\gamma = 1/(2\sigma^2)$):

$$G[n] = \sqrt{\frac{\gamma}{\pi}} e^{-\gamma n^2}, \tag{1}$$

$$G_{\mathrm{D}}[n] \equiv (-2\gamma n) G[n], \tag{2}$$

$$G_{\mathrm{DD}}[n] \equiv (4\gamma^2 n^2 - 2\gamma) G[n] \tag{3}$$

- $x[n]$: input signal defined in $[0, N]$.
- Gaussian smoothing and its differentials:

$$x_{\mathrm{G}}[n] \equiv \sum_{k=-K}^{K} G[k] x[n-k], \tag{4}$$

$$x_{\mathrm{GD}}[n] \equiv \sum_{k=-K}^{K} G_{\mathrm{D}}[k] x[n-k], \tag{5}$$

$$x_{\mathrm{GDD}}[n] \equiv \sum_{k=-K}^{K} G_{\mathrm{DD}}[k] x[n-k]. \tag{6}$$

- Because $G[n]$ decays rapidly for $|n| \geq 3\sigma$, they are calculated in an interval $[-K, K]$ for an integer K.

2.2 Gaussian smoothing and its differentials using SFT

- $\beta = \pi/K$.
- Definition of **SFT**:

$$c_p[n] = \sum_{k=-K}^{K} x[n-k]\cos(\beta pk), \tag{7}$$

$$s_p[n] = \sum_{k=-K}^{K} x[n-k]\sin(\beta pk). \tag{8}$$

  Note that the bases in this SFT are not orthogonal.
- Approximations of the Gaussian function and its differentials:

$$\hat{G}[n] \equiv \sum_{p=0}^{P} a_p \cos(\beta pn) \simeq G[n], \tag{9}$$

$$\hat{G}_{\mathrm{D}}[n] \equiv \sum_{p=1}^{P} b_p \sin(\beta pn) \simeq G_{\mathrm{D}}[n], \tag{10}$$

$$\hat{G}_{\mathrm{DD}}[n] \equiv \sum_{p=0}^{P} d_p \cos(\beta pn) \simeq G_{\mathrm{DD}}[n]. \tag{11}$$

- a_p, b_p, and d_p can be decided by the minimum mean square error (MMSE):

$$\sum_{n=-K}^{K} \left| \hat{G}[n] - G[n] \right|^2, \tag{12}$$

$$\sum_{n=-K}^{K} \left| \hat{G}_{\mathrm{D}}[n] - G_{\mathrm{D}}[n] \right|^2, \tag{13}$$

$$\sum_{n=-K}^{K} \left| \hat{G}_{\mathrm{DD}}[n] - G_{\mathrm{DD}}[n] \right|^2. \tag{14}$$

- The smoothed signals and its differentials are given by

$$x_{\mathrm{G}}[n] \simeq \sum_{p=0}^{P} a_p c_p[n], \tag{15}$$

$$x_{\mathrm{GD}}[n] \simeq \sum_{p=1}^{P} b_p s_p[n], \tag{16}$$

$$x_{\mathrm{GDD}}[n] \simeq \sum_{p=0}^{P} d_p c_p[n]. \tag{17}$$

2.3 SFT using kernel integrals

- **Kernel integral**:

$$u[n] = \sum_{k=0}^{n} x[k] e^{i\beta pk}.$$

- Recurrence formula:

$$u[n] = u[n-1] + x[n] e^{i\beta pn}.$$

- **SFT by kernel integral**:

$$c_p[n] - i s_p[n] = e^{-i\beta pn}(u[n] - u[n-2K-1]).$$

2.4 SFT using IIR filters

- **First-order IIR filter**:

$$v[n] = e^{-i\beta p} v[n-1] + x[n].$$

- **SFT by IIR filter**:

$$c_p[n] - i s_p[n] = (-1)^p (v[n+K] - v[n-K] + x[n-K])$$

- **Second-order IIR filter**:

$$v[n] = 2\cos(\beta p) v[n-1] - v[n-2] + x[n] - e^{i\beta p} x[n-1].$$

When either of $c_p[n]$ or $s_p[n]$ is necessary, we can calculate only real or imaginary part.

3 Stabilized calculation of Gaussian smoothing and its differentials using ASFT

3.1 ASFT

- $\alpha(>0)$: Decay parameter
- **Definition**:

$$\tilde{c}_p[n] = \sum_{k=-K}^{K} x[n-k] e^{\alpha k} \cos(\beta p k),$$

$$\tilde{s}_p[n] = \sum_{k=-K}^{K} x[n-k] e^{\alpha k} \sin(\beta p k).$$

- **First-order IIR filter for ASFT**:

$$\tilde{v}[n] = e^{-\alpha - i\beta p} \tilde{v}[n-1] + x[n].$$

- **ASFT by first-order IIR filter**:

$$\tilde{c}_p[n] - i\tilde{s}_p[n] = (-1)^p e^{-\alpha K} (\tilde{v}[n+K] - e^{-\alpha(2K)} \tilde{v}[n-K] + e^{-2\alpha K} x[n-K])).$$

- **Second-order IIR filter for ASFT**:

$$\tilde{v}[n] = 2e^{-\alpha} \cos(\beta p) \tilde{v}[n-1] - e^{-2\alpha} \tilde{v}[n-2] + x[n] - e^{-\alpha + i\beta p} x[n-1].$$

3.2 Gaussian smoothing and its differentials using ASFT

- Let we choose α such that $n_0 = \frac{\alpha}{2\gamma}$ becomes an integer.

$$e^{\alpha n} G[n] = e^{\frac{\alpha^2}{4\gamma}} G[n - n_0]$$

- Decay in Gaussian function is equivalent to pixel shift with a constant factor.
- Gaussian smoothing and its differentials:

$$\begin{aligned}
x_{\mathrm{G}}[n] \simeq & e^{-\frac{\alpha^2}{4\gamma}} \sum_{p=0}^{P} a_p \tilde{c}_p[n + n_0], \\
x_{\mathrm{GD}}[n] \simeq & e^{-\frac{\alpha^2}{4\gamma}} \sum_{p=0}^{P} \alpha a_p \tilde{c}_p[n + n_0] - \sum_{p=1}^{P} 2\gamma b_p \tilde{s}_p[n + n_0], \\
x_{\mathrm{GDD}}[n] \simeq & e^{-\frac{\alpha^2}{4\gamma}} \sum_{p=0}^{P} (d_p + \alpha^2 a_p) \tilde{c}_p[n + n_0] + \sum_{p=1}^{P} (2\alpha) \tilde{b}_p s_p[n + n_0].
\end{aligned}$$

4 Minimum mean square error with suppression of discontinuity at edges

- In order to **suppress the discontinuities at edges**, we propose simple criteria **MMSE-SDE** to calculate coefficients a_p, b_p, and d_p.
- They decrease heights at edge points:

$$\sum_{n=-K}^{K} \left|\hat{G}[n] - G[n]\right|^2 + \mu \left|\hat{G}[-K]\right|^2,$$

$$\sum_{n=-K}^{K} \left|\hat{G}_{\mathrm{D}}[n] - G_{\mathrm{D}}[n]\right|^2 + \mu \left|\hat{G}_{\mathrm{D}}[-K]\right|^2,$$

$$\sum_{n=-K}^{K} \left|\hat{G}_{\mathrm{DD}}[n] - G_{\mathrm{DD}}[n]\right|^2 + \mu \left|\hat{G}_{\mathrm{DD}}[-K]\right|^2.$$

- **Discontinuity measure**:

$$S_{\mathrm{G}} = \frac{\left|\hat{G}[-K]\right|}{\left|\hat{G}[-K] - \hat{G}[-K+1]\right|} \tag{18}$$

- μ is decided such that $S_{\mathrm{G}} \leq 1.1$.

5 Experimental results

- Abbreviations of filters

Abbrevi -ation	Type	Order	Block	SFT/ASFT
Direct	FIR	$6\sigma + 1$		
IIR1	IIR	1	$\times$	SFT
IIR1A	IIR	1	$\times$	ASFT
IIR1B	IIR	1	$\circ$	SFT
IIR1BA	IIR	1	$\circ$	ASFT
IIR2	IIR	2	$\times$	SFT
IIR2A	IIR	2	$\times$	ASFT
IIR2B	IIR	2	$\circ$	SFT
IIR2BA	IIR	2	$\circ$	ASFT
KI	Kernel Inte.		$\times$	
KIB	Kernel Inte.		$\circ$	

- 'Block': Not $v[n]$ but $v[n+K] - v[n-K]$ is calculated by an IIR filter.

5.1 Comparison of approximation error between SFT and ASFT

RMSE of Gaussian function and its differentials with coefficients by MMSE ($K = 256$ and $n_0 = 10$).

Transform	P	E_{G} (%)	E_{GD} (%)	E_{GDD} (%)	S_{G}
SFT	2	1.0	5.1	8.2	2239.5
	3	0.15	0.90	2.77	530.2
	4	0.038	0.24	0.54	603.2
	5	0.0059	0.043	0.16	224.4
	6	0.0015	0.011	0.031	291.7
ASFT	2	1.1	5.4	8.5	575.0
	3	0.17	1.02	3.10	272.5
	4	0.046	0.30	0.63	256.2
	5	0.017	0.037	0.12	138.5
	6	0.0021	0.016	0.041	149.2

- Definition of RMSE:

$$E_{\mathrm{G}} = \sqrt{\sum_{n=-3K}^{3K} \left|\hat{G}[n] - G[n]\right|^2 \Big/ \sum_{n=-3K}^{3K} |G[n]|^2}.$$

- RMSEs of ASFT increase by from 10% to 30% of RMSEs of SFT.
- E_{G} =0.46% by Direct with the interval of $\pm 3\sigma$

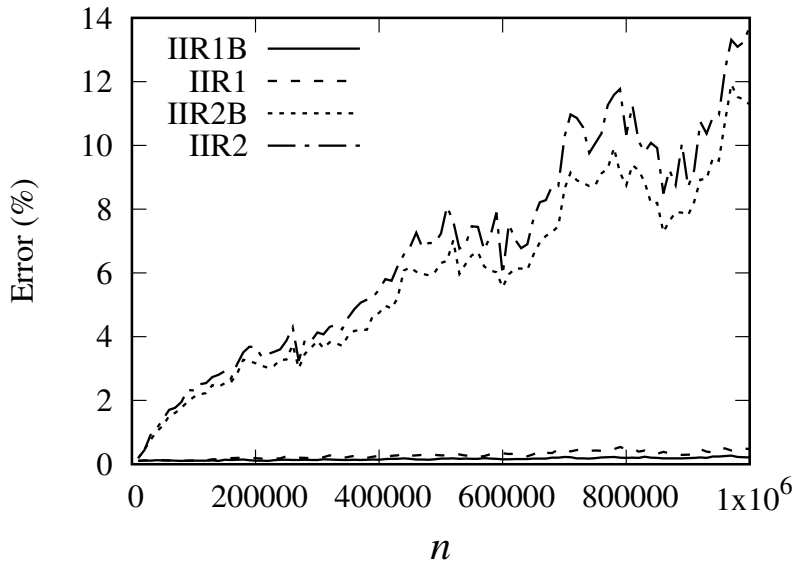
5.2 Error caused by using the single-precision floating-point calculation

- The standard image 'Barbara' is converted to an 1D data of length 262144 and we repeat it four times so that the length of the input data is 1048576.
- $x_G[n]$: Directly calculated result of the Gaussian smoothing with enough long interval
- $\hat{x}_G[n]$ be approximated result.
- Error at n is evaluated by

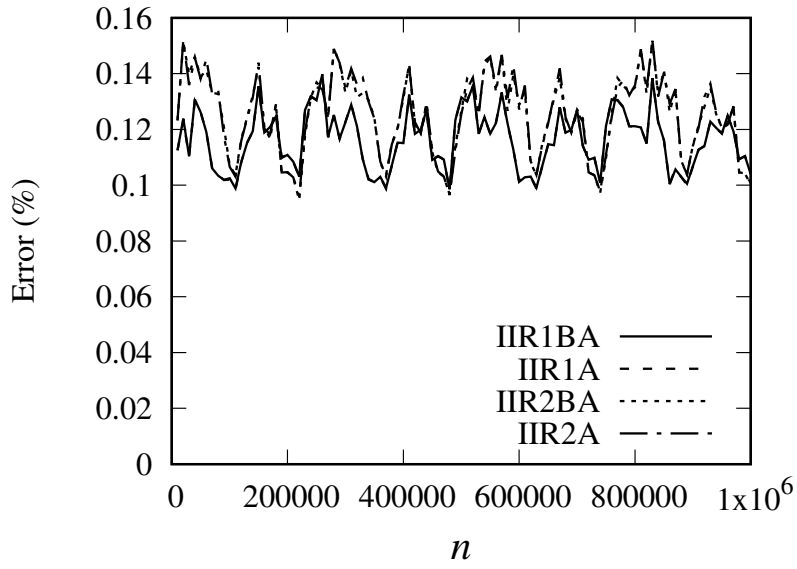
$$\frac{\max_{n \le k < n+10000} |\hat{x}_G[k] - x_G[k]|}{\sqrt{\frac{1}{10000} \sum_{n}^{n+10000} |x_G[k]|^2}} \tag{19}$$

- Results: (next page)

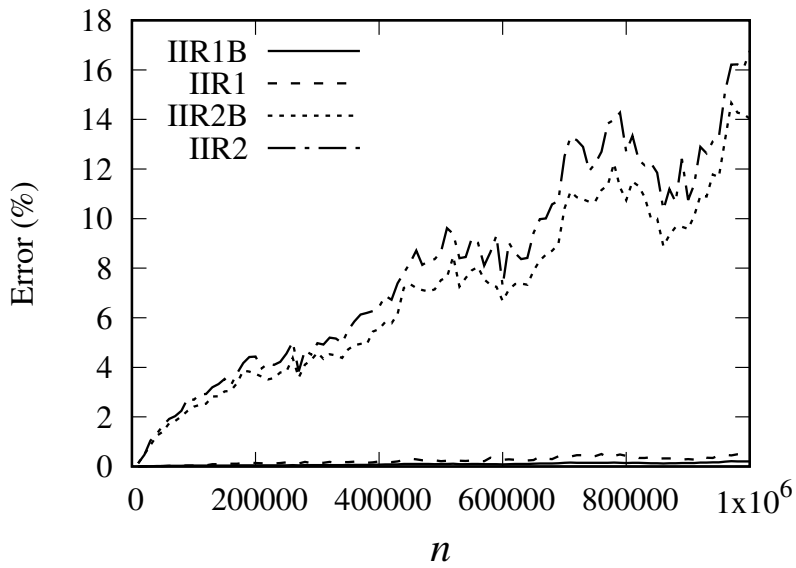
- Summary of results:
  - Errors of smoothing by **second-order filter are large**.
  - Errors of smoothing by ASFT are **small and not be accumulated**.



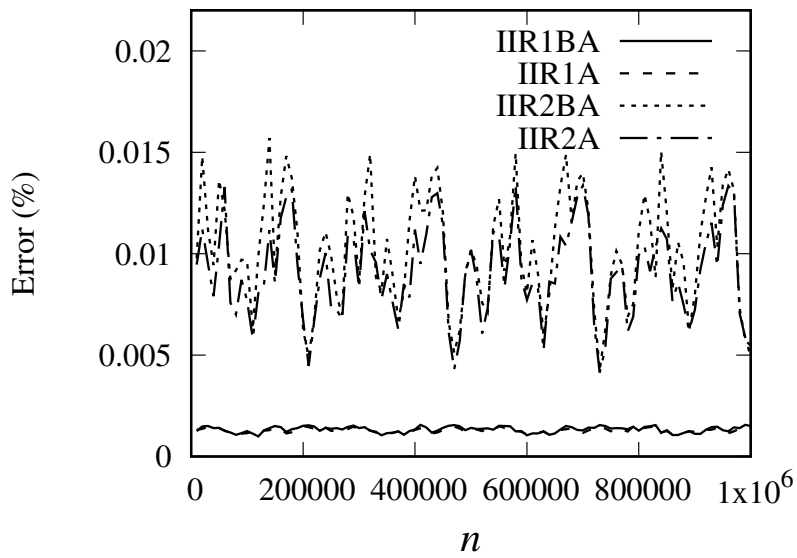
(a) SFT ($P = 3$)



(b) ASFT ($P = 3$)



(c) SFT ($P = 6$)



(d) ASFT ($P = 6$)

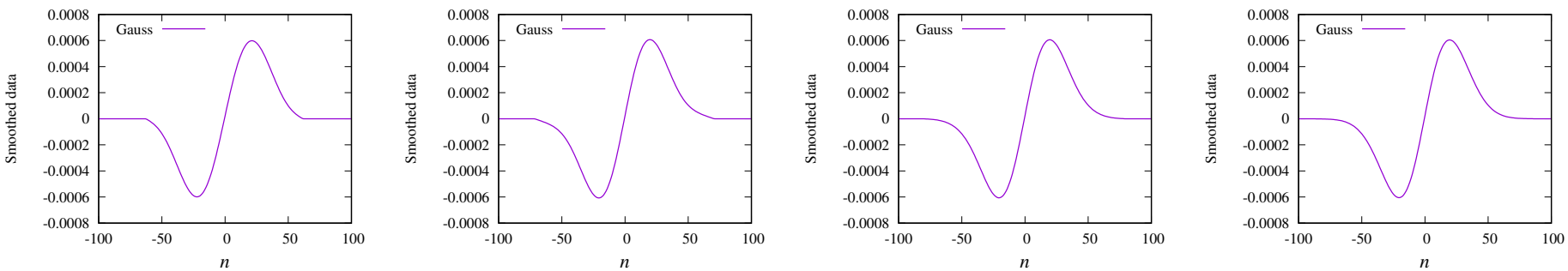
RMSE by single precision floating-point calculation

5.3 MSE of Gaussian smoothing and its differentials with coefficients by MMSE-SDE

RMSE of Gaussian function and its differentials with coefficients by MMSE-SDE ($K = 256$ and $n_0 = 10$)

Transform	P	E_{G} (%)	E_{GD} (%)	E_{GDD} (%)	S_{G}
SFT	2	2.4	5.1	13.3	0.973
	3	0.31	0.90	5.62	1.07
	4	0.087	0.24	1.02	0.85
	5	0.0119	0.043	0.32	0.91
	6	0.0030	0.011	0.052	0.90
ASFT	2	2.6	5.3	14.0	0.97
	3	0.36	1.01	6.27	1.07
	4	0.11	0.29	1.24	0.84
	5	0.017	0.057	0.42	0.90
	6	0.0044	0.015	0.076	0.89

- E_{G} and E_{GDD} of MMSE-SDE are about **twice** comparing to those of MMSE.
- S_{G} is **much decreased** by MMSE-SDE.



(a) $P = 2$ (b) $P = 3$ (c) $P = 4$ (d) $P = 6$

Gaussian smoothing by coefficients based on MMSE-SDE

5.4 Calculation time

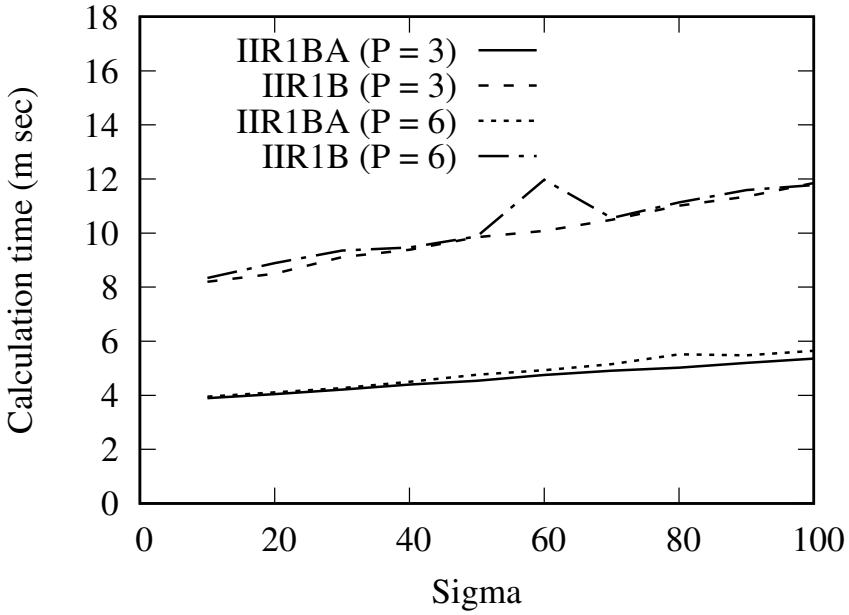
- Image size: 512×512 (Barbara).
- Intel Core CPU i5-6400 @3.3 GHz with 8 GB RAM.

Calculation time ($P = 3$, $K = 256$, and $n_0 = 3$)

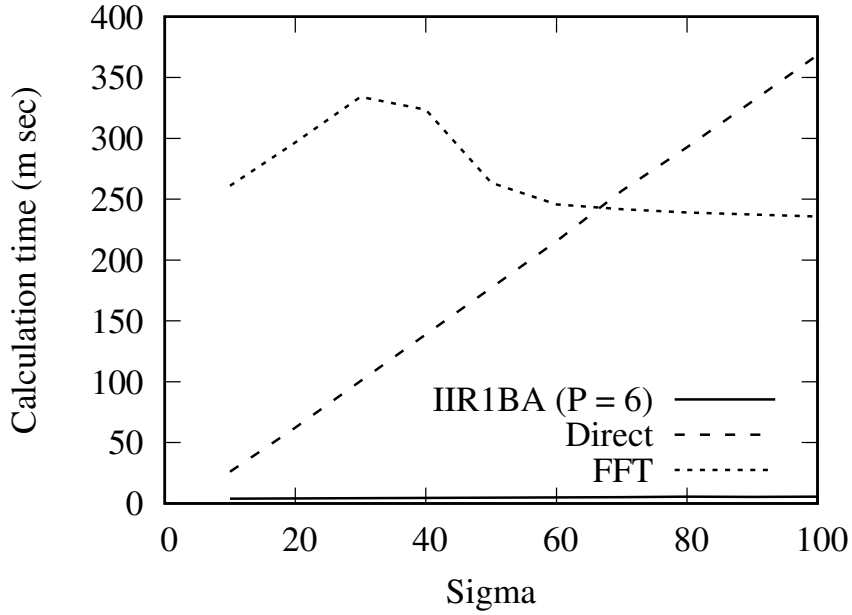
Method Method	Smoothing (ms)	Derivative (ms)	Laplacian (ms)
Direct	62.5	99.0	98.3
FFT	306.4	536.1	308.8
IIR1	8.6	14.5	12.4
IIR1A	8.6	14.2	12.6
IIR1B	4.1	8.9	9.0
IIR1BA	4.1	7.9	7.7
IIR2	8.9	15.5	13.2
IIR2A	8.6	16.1	15.2
IIR2B	8.2	11.6	12.1
IIR2BA	8.6	15.4	14.1
KI	10.3	16.6	15.2
KIB	12.6	22.7	18.3

Calculation time ($P = 6$, $K = 256$, and $n_0 = 3$)

Method Method	Smoothing (ms)	Derivative (ms)	Laplacian (ms)
Direct	62.5	99.0	98.3
FFT	306.4	536.1	308.8
IIR1	9.8	15.9	15.4
IIR1A	10.2	16.8	16.4
IIR1B	8.9	14.3	13.6
IIR1BA	8.7	14.5	13.7
IIR2	9.9	17.9	14.7
IIR2A	9.8	22.6	21.9
IIR2B	8.6	15.3	13.0
IIR2BA	8.6	15.3	12.9
KI	13.2	21.0	19.7
KIB	16.0	28.5	25.4



(c) SFT and ASFT



(d) ASFT, ASFT, Direct, and FFT

Calculation time ($n_0 = 3$)

- Calculation times by SFT/ASFT are almost constant against σ.
- Calculation times by direct calculation are proportional to σ.
- Calculation times by FFT are almost constant against σ but long.

6 Conclusion

- We explained the conventional methods of SFT for calculating Gaussian smoothing and its differentials, and clarified two problems:
  - Accumulation of errors in case of single-precision floating-point calculation
  - Discontinuities at edges of SFT intervals
- We proposed
  - Attenuated SFT (ASFT) featuring a decay factor
  - A new criterion for efficiently reducing discontinuities at the edges of SFT intervals.
- Experimental results showed the marked advantages of the proposed methods.
- For future works, it is promising to implement the proposed algorithms on GPU and apply them to practical tasks of image processing and computer vision.

All source codes including **mex code for matlab** are obtained from `https://github.com/heavenstime/gaussSmooth`