Signature features with the visibility transformation

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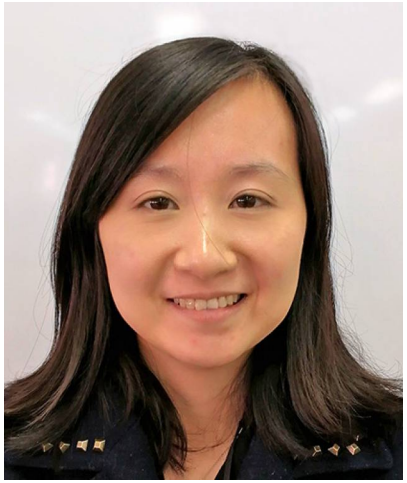
DataSig

A rough path between
mathematics and data science

Co-authors and acknowledgements



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Outline

1. Signatures as a natural feature set

2. Visibility transformation

Signatures as a natural feature set

A sequence of (input,output) pair, denoted by $\{(\mathbf{X}_i, y_i)\}_i$, we make inference f between input and output such that

$$f(\mathbf{X}_i) \approx y_i.$$

Signatures of path (data stream) as features:

- Determining solutions for controlled differential equations [Chen, 1958], [Hambly&Lyons, 2010], [Boedihardjo et al, 2016].
- Invariance under time reparameterisation.
- Unwrapping nonliearity.
- Fixed dimension under length variations, vectorisation.
- Faithful representation: tree-like equivalence.

Definition: signatures

Consider time-dependent path $X : [a, b] \to \mathbb{R}^d$. Now for any $t \in [a, b]$ and a fixed ordered multi-index collection $(i_1, \ldots, i_k)$, with $k \in \mathbb{N}$ and $i_j \in [d]$ for $j \in [k] := \{1, \ldots, k\}$, define the coordinate iterated integral by

$$S(X)_{a,t}^{i_1,\ldots,i_k} := \int_{a<t_k<t} \cdots \int_{a<t_1<t_2} \mathrm{d}X_{t_1}^{i_1} \ldots \mathrm{d}X_{t_k}^{i_k}. \tag{1}$$

Definition

The *signature* of a path X is the infinite collection of all iterated integrals of X:

$$S(X)_{a,b} := (1, S(X)_{a,b}^1, \ldots, S(X)_{a,b}^d, S(X)_{a,b}^{1,1}, S(X)_{a,b}^{1,2}, \ldots). \tag{2}$$

The finite collection of all terms $S(X)_{a,b}^{i_1,\ldots,i_k}$ with the multi-index of fixed length k is termed as the *kth level of the signature*. The truncated signature up to the pth level is denoted by $\lfloor S(X)_{a,b} \rfloor_p$.

Only capturing the effect of pattern change and not ones depending on the absolute position.

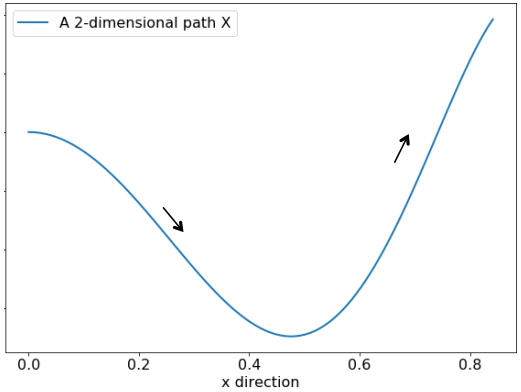
$$\text{Path X} \overset{\text{Signature}}{\Longrightarrow} \text{Incremental effects of path X.}$$

Outline

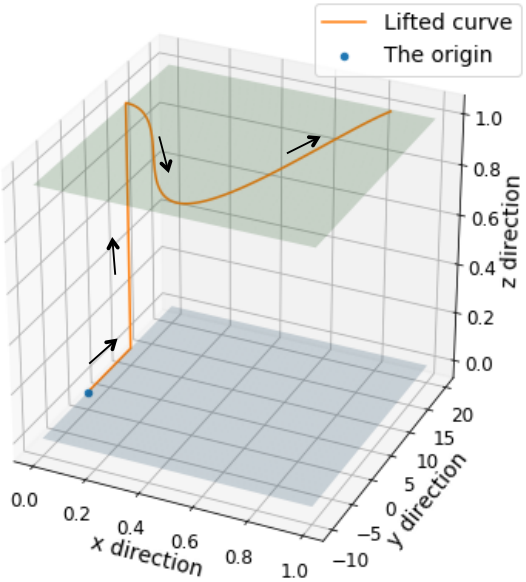
1. Signatures as a natural feature set

2. Visibility transformation

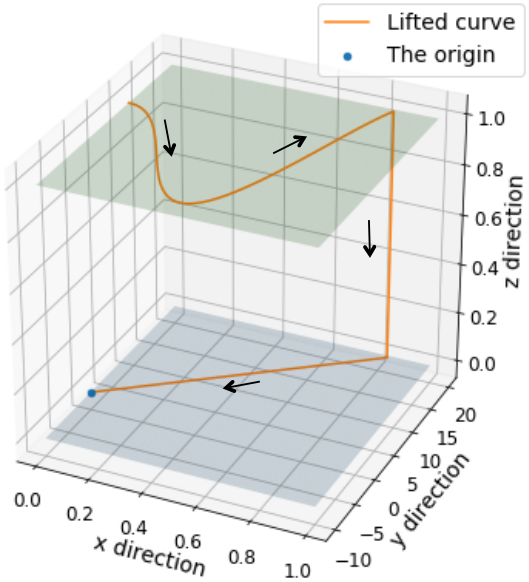
For continuous path of finite length



(a) Original 2-dimensional path X.



(b) $\mathbf{I}$-visibility transformation ($\gamma_{\mathbf{I}}$).



(c) $\mathbf{T}$-visibility transformation ($\gamma_{\mathbf{T}}$).

Main property

Theorem

Given X and a multi-index collection J with $d+1 \notin J$. Define $J^- := (d+1|J)$ and $J^+ := (J|d+1)$, where $d+1$ is prefixed (resp. postfixed) to J on the left(resp. right). Then

$$S(\gamma_{\mathbf{I}}(X))^{J^-} = S_X^J e_{d+1} e_{j_1} \dots e_{j_{|J|}},\ S(\gamma_{\mathbf{I}}(X))^{J^+} = \frac{1}{|J|!} \prod_{j \in J} X_0^j e_{j_1} \cdots e_{j_{|J|}} e_{d+1}, \tag{3}$$

where S_X^J is the corresponding coefficient of $S(X)$. Similarly, for the $\mathbf{T}$-visibility transformation, we have

$$S(\gamma_{\mathbf{T}}(X))^{J^+} = S_X^J e_{j_1} \dots e_{j_{|J|}} e_{d+1},\ S(\gamma_{\mathbf{T}}(X))^{J^-} = \frac{(-1)^{|J|+1}}{|J|!} \prod_{j \in J} X_1^j e_{d+1} e_{j_1} \cdots e_{j_{|J|}}. \tag{4}$$

Path X $\overset{\text{VT}}{\Longrightarrow}$ Path $\bar{\text{X}}$ $\overset{\text{Signature}}{\Longrightarrow}$ Positional and incremental effects of path X

discrete VT

Given streamed data $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, $\mathbf{x}_i \in \mathbb{R}^d$.

$$\text{data } \mathbf{X} \xRightarrow[\text{linear interpolation}]{\text{piecewise}} \text{path } X \Rightarrow \text{signatures}$$

$$\text{data } \mathbf{X} \xRightarrow{\text{Python packages}} \text{signatures}$$

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One needs a discrete transform such that

$$\text{data } \mathbf{X} \xRightarrow{\text{discrete T}} \text{data } \bar{\mathbf{X}} \xRightarrow{\text{Python packages}} \text{signatures}$$

and

$$\text{data } \bar{\mathbf{X}} \xRightarrow[\text{linear interpolation}]{\text{piecewise}} \text{path } \bar{X}$$

Discrete VT

Take a discrete data with two 2-dimensional observations $[1,2]^T, [3,4]^T$ for example, the discrete $\mathbf{I}$-visibility transformation (IVT) would give

$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix} \overset{\text{IVT}}{\Longrightarrow} \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix},$$

and the discrete $\mathbf{T}$-visibility transformation (TVT) would give

$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix} \overset{\text{TVT}}{\Longrightarrow} \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Discrete VT

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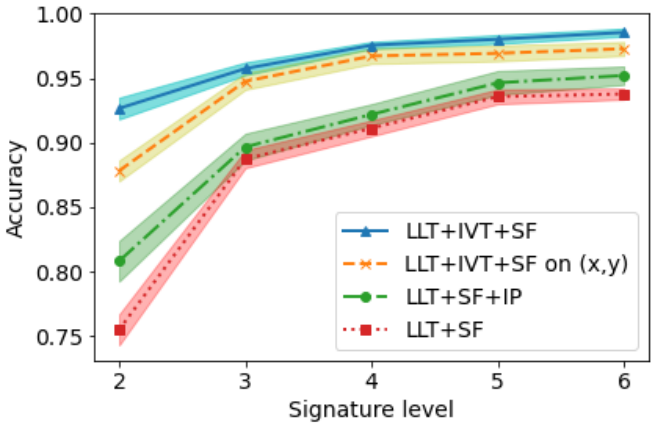
Application: character trajectories dataset

- The data [Williams et al, 2006] consists of 2858 instances for 20 different characters, and was captured using a WACOM tablet at 200Hz. Each character sample is a 3-dimensional pen tip velocity trajectory, namely (x, y, p). The original handwriting data contains training set (50%) and testing set (50%).

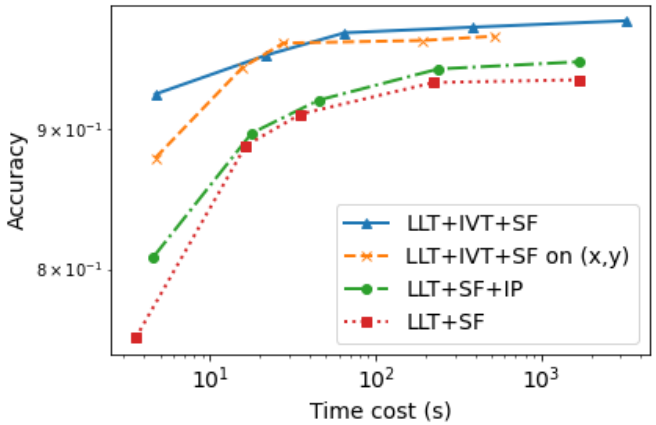
- 4 Different features: truncated signatures with the lead lag transform (LLT), truncated signatures with LLT and being prefixed by the explicit initial position (LLT+IP), and truncated signatures with LLT and the discrete $\mathbf{I}$-visibility transformation (LLT+IVT). We also extracted truncated signature features with LLT and IVT on the trajectory, namely the (x, y) path (LLT+IVT on (x, y)). In the experiment, the signature features were truncated to levels $\{2, 3, 4, 5, 6\}$.

- lightGBM Classifer: hyperparameter tuning implemented via grid search with cross validation.

Application: character trajectories dataset



(a) Signature level VS average accuracy with standard deviation.

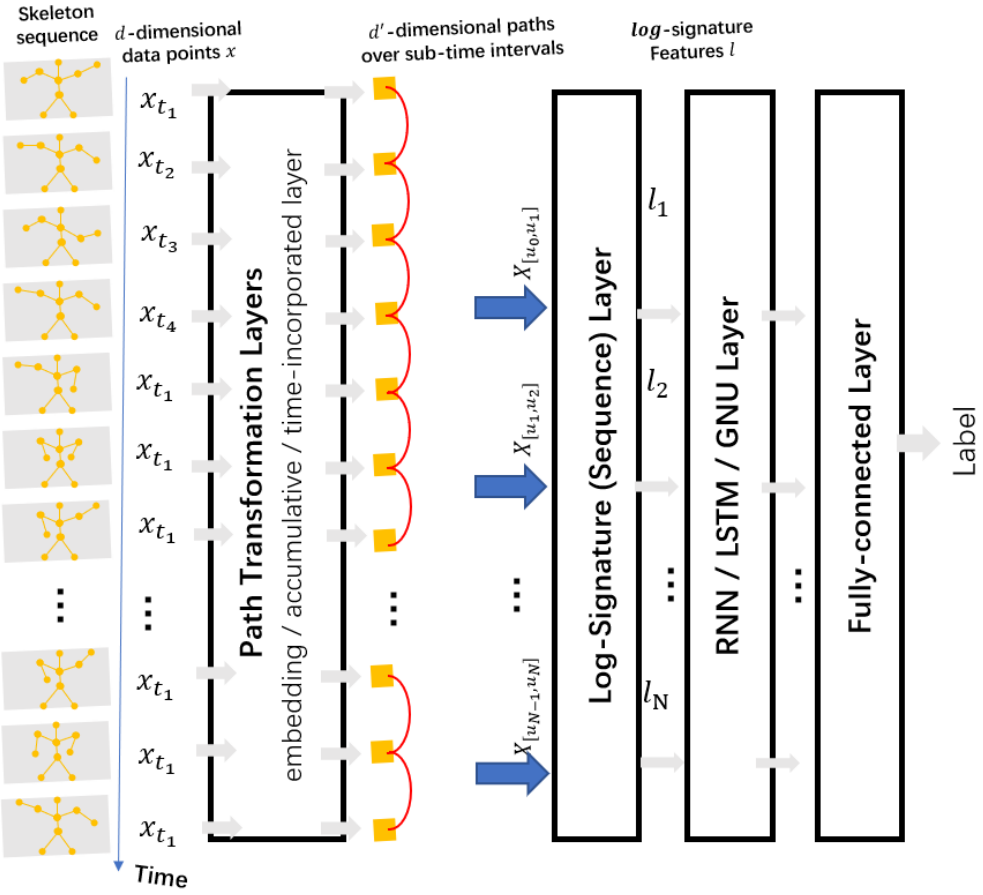


(b) CPU time VS average accuracy (log-log scale).

Method	Accuracy
ϕ(O,HMM)+SVM [Perina et al, 2009]	92.91%
TK [Tsuda et al, 2002]	93.67%
LLT+IVT+SF on (x, y)	**97.27%**
SDD [grabocka et al, 2016]	98.00%
MCDS [iosifidis et al, 2012]	98.25%
LLT+IVT+SF	**98.54%**

Gesture classification: Chalearn 2013 data

- The ChaLearn 2013 multi-modal gesture dataset [Escalera et al,2013] contains 23 hours of Kinect data of 27 subjects performing 20 Italian gestures.
- Liao et al [Liao et al, 2019] proposed a log-signature-based recurrent neural network model.



Gesture classification: Chalearn 2013 data

Modified PT-Logsig-RNN: adding a visibility transformation layer to architecture of PT-Logsig-RNN Model.

Table: Comparison of different methods on the Chalearn 2013 data.

Method	Accuracy
Deep LSTM [Shahroudy et al, 2016]	87.10%
Two-stream LSTM [Wang et al, 2017]	91.70%
ST-LSTM + Trust Gate [Liu et al, 2017]	92.00%
Three-stream net TTM [Li et al, 2019]	92.08%
PT-Logsig-RNN [Liao et al, 2019]	92.21%
Modified PT-Logsig-RNN	**92.89%**

Conclusion

To capture the informative information from streamed data for learning tasks, we explore a transformation that encodes the effects on the absolute position of streamed data into signature features with theoretical justifications. The enhanced feature is unified, theoretical-backed, and simple to implement with. It is superior to many benchmark methods that require handy data preparation and implementation of complicated algorithms in applications when absolute position of the data is intrusive.

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