# Stochastic Runge-Kutta methods and adaptive SGD-G2 stochastic gradient descent ICPR 2020 paper # 2258

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# Adaptive Stochastic Gradient Descent

• training neural networks (NN) e.g., for classification to simplify, often relies on the minimization of a loss function : $f(X) := \frac{1}{N}\sum_{i=1}^{N} f(\omega_i, X)$, where the sum is over all available samples. Equivalent writing: $f(X) = \mathbb{E}_\omega f(\omega, X)$. $X \in \mathbb{R}^d$ = parameters of the NN.

• classical gradient descent procedure : $X_{n+1} = X_n - h\nabla f(X_n)$, $h > 0$ is the learning rate (="step size").

• BUT computing $\nabla f(X_n)$ is too costly because of the average (many samples).

• it is replaced by a crude approximation $X_{n+1} = X_n - h\nabla f(\omega_{\gamma_n}, X_n)$ where $(\gamma_n)_{n\geq 1}$ are i.i.d uniform random variables in $\{1, 2, .., N\}$.
This is the Stochastic Gradient Descent (SGD)

# Adaptive Stochastic Gradient Descent

- Stochastic Gradient Descent $X_{n+1} = X_n - h\nabla f(\omega_{\gamma_n}, X_n)$

$(\gamma_n)_{n\geq 1}$ are i.i.d uniform in $\{1, 2, .., N\}$.
Problem: small $h$ converge slowly, large $h$ unstable.

- MAIN QUESTION: how to (optimally) choose the learning rate (l.r.) $h$ ?

- Flow interpretation : in the limit $h \to 0$ the minimization of $f(X)$ is some approximation of the 'continuous time' evolution equation $X'(t) = \nabla_X f(X(t))$. SGD: $X_n \simeq X(t_n)$, $t_n = n \cdot h$.

- MAIN IDEA:

1/ construct a better approximation $Y_{n+1}$ of $X(t_{n+1})$ such that $Y_{n+1} - X_{n+1}$ is an estimation of the error $X_{n+1} - X(t_{n+1})$.
2/ Using $Y_{n+1}$ compute the largest l.r. $h$ such that stability still holds

- Question 1: find a high order scheme consistent for the flow dynamics
- Question 2: is the procedure performing well in practice..

# The second order Stochastic Runge Kutta "SRK" scheme

## Stochastic Runge Kutta (SRK)

$$\tilde{Y}_{n+1} = Y_n - h\nabla f_{\gamma_n}(Y_n),\ \ Y_{n+1} = Y_n - \frac{h}{2}\left[\nabla f_{\gamma_n}(Y_n) + \nabla f_{\gamma_n}(\tilde{Y}_{n+1})\right]. \quad (1)$$

**Theorem (Convergence of SGD and SRK schemes, I.A., G.T. 2019)**

*Suppose* $\forall k$, $\nabla f_k$ *is a Lipschitz function,* $\nabla f_k$ *and its partial derivatives up to order* 6 *have at most polynomial increase at* $\infty$ *and* $\nabla f_k$ *increases at most linearly at infinity. Then the SGD scheme converges at (weak) order* 1 *(in h) while the SRK scheme* (1) *converges at (weak) order* 2*.*

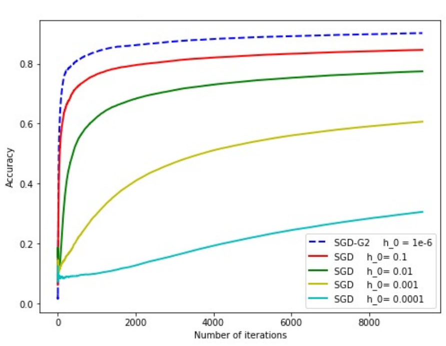
# Adaptive step SGD: the SGD-G2 algorithm

**Algorithm 1** SGD-G2

**Set hyper-parameter:** $\beta$, mini-batch size $M$, choose stopping criterion
**Input:** initial learning rate $h_0$, initial guess $X_0$
**Initialize iteration counter:** $n = 0$
**while** stopping criterion not met **do**
  select next mini-batch $\gamma_n^m$, $m = 1, ..., M$
  Compute $g_n = \frac{1}{M}\sum_{m=1}^{M} \nabla f_{\gamma_n^m}(X_n)$
  Compute $\tilde{g}_n = \frac{1}{M}\sum_{m=1}^{M} \nabla f_{\gamma_n^m}(X_n - h_n g_n)$
  Compute $h_n^{opt} = \begin{cases} \frac{3}{2}\frac{h_n\langle g_n - \tilde{g}_n, g_n\rangle}{\|g_n - \tilde{g}_n\|^2} & \text{if } \langle g_n - \tilde{g}_n, g_n\rangle > 0 \\ h_n & \text{otherwise.} \end{cases}$
  **if** $h_n^{opt} > h_n$ **then**
    $h_{n+1} = \beta h_n + (1-\beta) h_n^{opt}$
  **else**
    $h_{n+1} = h_n^{opt}$
  **end if**
  Update $X_{n+1} = X_n - h_{n+1} g_n$
  Update $n \to n + 1$
**end while**

# Empirical validation (MNIST / FMNIST / CIFAR10)

Results on standard datasets are very convincing, start with *h* small then let it adapt itself.



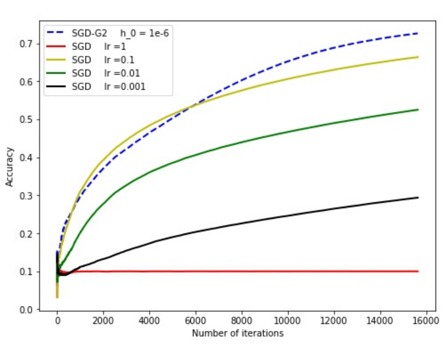




Figure: **Left:** SGD vs. SGD-G2 on FMNIST . **Right:** SGD vs. SGD-G2 on CIFAR10 (10 epochs).

# Empirical validation on CIFAR100

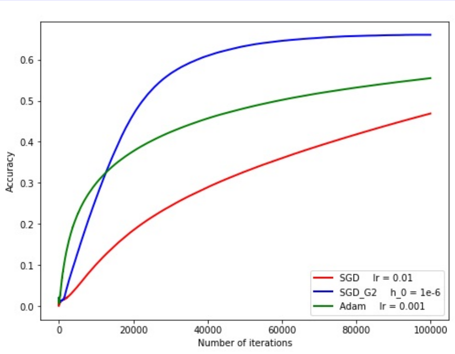



Figure: SGD , SGD-G2 and Adam (100 epochs) on CIFAR100.

# Conclusion

• We presented a new adaptive learning rate procedure that performs well on standard datasets (MNIST, FMNIST, CIFAR10, CIFAR100)

• in the process we came up with a proof for the convergence of the Stochastic Runge-Kutta second order scheme

Want to know more:

• the paper: https://arxiv.org/abs/2002.09304 (Arxiv ID= arXiv:2002.09304)

• these slides: https://doi.org/10.5281/zenodo.4314299 (DOI=10.5281/zenodo.4314299)

• this video: https://youtu.be/z_V2OIM0UmI